Exam: September 26, 2014

Ανάλυση Αβεβαιοτήτων σε Προσομοιώσεις Μηχανολογικών Συστημάτων (Uncertainty Quantification in Engineering Science)

Final Exam

Duration: 5 hours

Open books and notes

Problem 1: (20 points)

The posterior distribution of the parameters of a model is given by

$$p(\theta_1, \theta_2 \mid D, I) \propto \exp\left[-\frac{1}{2}\left(\theta_1^2 + \theta_2^2 + 2\mu\theta_1\theta_2 - 2\mu\theta_1 - 2\theta_2\right)\right]$$

Find the uncertainty region and plot it in the two-dimensional parameter space (θ_1, θ_2).

Hint: Need to find the most probable point, the Hessian, the covariance matrix and then clearly plot the contour plots of the posterior distribution in the two-dimensional parameter space, indicate the principal direction of the ellipsoid, as well as the length of the uncertainty along the principal axes of the ellipsoid.

Problem 2: (25 points)

It is given a model with output quantity of interest

$$y(t) = Ae^{\theta_1 t} + B\theta_2 + E$$

where E is an error term arising from the model error. The values of A and B are given, while the parameters θ_1 and θ_2 are considered uncertain and independent. The error term E is Gaussian with zero mean and variance s^2, i.e. $E \sim N(0, s^2)$. Assuming that the uncertain parameter vector $\underline{\theta} = (\theta_1, \theta_2)^T$ follows a Gaussian distribution with mean $\underline{\mu} = (\mu_1, \mu_2)^T$ and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

, our problem is to estimate the uncertainty in the response quantity of interest $y(t)$ as a function of time t. Specifically, find the mean and the variance of $y(t)$ in terms of A, B, μ_1, μ_2, σ_1, σ_2, s and t.

Problem 3: (20 points)

The posterior probability density function of a set of two parameters $\underline{\theta} = (\theta_1, \theta_2)^T$ is Gaussian with mean $\underline{0}$ and diagonal covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

Let $\underline{\theta}^{(j)}$ be the current sample in the Markov Chain Monte Carlo algorithm generated using a Metropolis-Hasting algorithm. Following Metropolis-Hasting algorithm, let $\underline{\xi}$ be the candidate sample drawn from a uniform distribution centered at the current sample $\underline{\theta}^{(j)}$. Let $\underline{\theta}^{(j)} = (1,0)^T$. If $\underline{\xi} \sim U([0,3],[0,1])$, is drawn from a uniform distribution with bounds $[0,1]$ for the first component ξ_1 and $[0,2]$ for the first component ξ_2

1. find the probability that the next sample in the chain will be $\underline{\theta}^{(j+1)} = \underline{\xi} = (0,1)^T$
2. find the probability that the next sample in the chain will be $\underline{\theta}^{(j+1)} = \underline{\xi} = (0,3)^T$
3. find the probability that the next sample in the chain will be $\underline{\theta}^{(j+1)} = \underline{\xi} = (3,0)^T$

Problem 4: (35 points)

Inference of Acceleration of Gravity and Air Resistance Coefficient for a Falling Object

Consider the mathematical model of a falling object with mass m, acceleration of gravity $g = 9.81 m/s^2$ and air resistance force $F_{res} = -m\beta\upsilon^2$, where β is the air resistance coefficient. Using Newton's law, the equation of motion of the falling object is

$$m\frac{d\upsilon(t)}{dt} = mg - m\beta\upsilon^2(t) \tag{1}$$

or equivalently

$$a(t) = g - \beta\upsilon^2(t)$$

Measurements for the acceleration and the velocity of the falling object are obtained at regular time intervals $k\Delta t$. The acceleration measurements are denoted by $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N) \equiv \{\hat{a}_k\}_{1\to N}$ and the corresponding velocity measurements are denoted by $(\hat{\upsilon}_1, \hat{\upsilon}_2, \ldots, \hat{\upsilon}_N) \equiv \{\hat{\upsilon}_k\}_{1\to N}$. Given the observation data $D \equiv (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_N, \hat{\upsilon}_1, \hat{\upsilon}_2, \ldots, \hat{\upsilon}_N)$ of the acceleration and velocity of the falling object at time instances $t = \Delta t, 2\Delta t, \ldots, N\Delta t$, respectively, we are interesting in estimating the uncertainty of the parameter β of the system. Note that the measurements and the model predictions satisfy the model error equation

$$\hat{a}_k = g - \beta\hat{\upsilon}_k^2 + E_k \tag{2}$$

$k = 1, \ldots, N$, where the measurement error terms E_k are independent identically distributed (iid) and follow a zero-mean Gaussian distribution $E_k \sim N(0,\sigma^2)$. The value of the variance σ^2 is given.

Assume a uniform prior for the parameter β and derive the expressions for the

1. Posterior PDF $p(\beta \mid D, \sigma, I)$.
2. The function $L(\beta) = -\ln p(\beta \mid D, \sigma, I)$
3. The MPV (or best estimate) $\hat{\beta}$ of β
4. The uncertainty in the parameter β

5. Derive the Gaussian asymptotic approximation for the posterior PDF of $p(\beta | D, \sigma, I)$. Is the Gaussian representation of the posterior uncertainty exact or approximate for this case?
6. Find the minimum number of data points required so that the uncertainty in β is less that a given value λ.
7. Find the uncertainty in the resistance force $F_{res} = -m\beta\upsilon^2$ given the uncertainties in the parameter β:
    a. Compute the mean of F_{res}
    b. Compute the standard deviation of F_{res}
    c. Find the probability density function that describes the uncertainty in F_{res}