Exercises – [Homework Assignment 2]

[Need only to solve problems 1 and 2]

Problem 1: Marginal distributions and uncertainty

The posterior uncertainty in two parameters x_1 and x_2 is found to be Gaussian with mean $\hat{\underline{x}} = (3,3)^T$ and covariance matrix $C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, where $-1 < \rho < 1$

1. Plot the spread of uncertainty around the best estimate $\hat{\underline{x}}$ (that is, plot a contour plot corresponding to $Q(\underline{x}) = 1$), for values of $\rho = 0, 0.1, 0.5, 0.9$

   Hint: Solve the eigenvalue problem $C\underline{\upsilon} = \mu\underline{\upsilon}$ and use the results in class to draw the contour plots. Note that $H^{-1} = C$ and that $\underline{u}$ and λ obtained from the eigenvalue problem $H\underline{u} = \lambda\underline{u}$ developed in class are related to $\underline{\upsilon}$ and μ as follows:

   $$\underline{\upsilon} = \underline{u}$$

   $$\mu = \frac{1}{\lambda}$$

2. Also, estimate the uncertainty in the marginal distribution of x_1 or x_2. Can the uncertainty in the marginal distribution of x_1 or x_2 describe the spread of uncertainty in the two dimensional space (x_1, x_2) of the two parameters?

Problem 2: Application to data fitting

Consider a set of data $D = (y_k, x_k), k = 1, \ldots, N$. Given this set of data D, use a linear model

$$y = F(x; \underline{a}) = a_1 x + a_0 \quad (1)$$

in order to describe the data, where $a = (a_0, a_1)$ is the unknown parameter set to be estimated. To account for model and measurement errors, assume the prediction error equation

$$y_k = F(x_k; \underline{a}) + e_k = a_1 x_k + a_o + e_k \quad (2)$$

where the prediction errors e_k are assumed to be i.i.d Gaussian with $e_k \sim N(0, \sigma^2)$, where σ^2 is unknown. Assuming uniform priors, with large enough bounds, find:

1. The posterior distribution of the model and prediction error parameters $\{a_0, a_1, \sigma^2\}$
2. The most probable value of the parameter set $\{a_0, a_1, \sigma^2\}$
3. The spread of uncertainty about the best estimate in the parameter space
4. The asymptotic estimate of the posterior distribution
5. The marginal posterior distribution of $\{a_0, a_1\}$.

6. Let $z = \gamma y + \eta$ be a relation between an output QoI and the measured quantity y, with $\eta \sim N(0, s^2)$ and γ, s^2 are known. Quantify the uncertainty on z at a new position x_0 given the uncertainties in the model parameters $\left\{a_0, a_1, \sigma^2\right\}$ by computing the distribution $p(z \mid D, I)$.

The uniform prior distribution is $\pi(\underline{a}, \sigma^2 \mid I) = \text{const}$, $\underline{a}_{\min} \le \underline{a} \le \underline{a}_{\max}$, $0 \le \sigma^2 \le \sigma^2_{\max}$ with very large bounds of the support of the uniform PDF.

Repeat the steps 1-6 assuming a Gaussian prior PDF $p(\underline{a} \mid I) = N(\underline{a} \mid \underline{\mu}_\pi, \Sigma_\pi)$.

Problem 3: Single DOF Mechanical Oscilator

Consider the mathematical model of an oscillator, with equation of motion given by

$$\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y = \frac{1}{m}F(t)$$

Assume the prediction error equation

$$\hat{Y}_k = y(t_k) + e_k$$

The prediction errors e_k are assumed to be i.i.d Gaussian variables with $e_k \sim N(0, \sigma^2)$, where σ^2 is unknown.

Let $F(t) = 0$ and $y(0) = y_0$, $\dot{y}(0) = \upsilon_0$ be the initial conditions. Assume that the mass of the oscillator and the initial conditions are given. Given a set of independent observations/data $D \equiv (\hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_N) \equiv \{\hat{Y}_k\}_{1 \to N}$, we are interesting in estimating the uncertainty (best estimate and co-variance)

A. in the modal frequency ω_0 and damping ratio ζ of the model given the initial conditions y_0 and υ_0.

B. in the initial conditions y_0 and υ_0 given the modal frequency ω_0 and damping ratio ζ of the model.

C. the modal frequency ω_0 and damping ratio ζ of the model and the initial conditions y_0 and υ_0.

Proceed and solve only B.

[Recognize that this is a special case of the general model fitting problem solved in class]

Assume uniform priors, with large enough bounds.

1. The posterior distribution of the parameters $\left\{y_0, \upsilon_0, \sigma^2\right\}$
2. The most probable value of the parameter set $\left\{y_0, \upsilon_0, \sigma^2\right\}$
3. The asymptotic estimate of the posterior distribution
4. The posterior distribution of $\left\{y_0, \upsilon_0\right\}$.
5. Let $R = \kappa y + \eta$ be a relation between an output QoI (the restoring force in the mechanical system) and the measured quantity y, $\eta \sim N(0, s^2)$, with κ and s^2 are known. Find the distribution $p(R \mid D, I)$ of R. Describe the uncertainty in the prediction R at a time instant t given the uncertainties in the model parameters $\left\{y_0, \upsilon_0, \sigma^2\right\}$.